

**MICROSCOPIC INTERPOLATION FORMULA FOR THE  
RIEMANN  $Z(t)$ -FUNCTION AND NEW ALGORITHM FOR  
ASYMPTOTIC SOLUTION OF SOME DIOPHANTINE  
EQUATION WITH LEIBNITZ COEFFICIENTS**

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ABSTRACT. In this paper we use the values of the Riemann  $Z(t)$ -function in order to construct certain quasi-orthonormal system of vectors. On this basis we prove a formula for microscopic interpolation of the function  $Z(t)$ . Simultaneously we have obtained a new algorithm how to construct asymptotic solutions of Diophantine equation with Leibnitz coefficients. This paper is English version of the paper [3] except the Appendix.

## 1. INTRODUCTION

Let (see [4], p. 79)

$$(1.1) \quad \begin{aligned} Z(t) &= e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right), \\ \vartheta(t) &= -\frac{t}{2} \ln \pi + \operatorname{Im} \Gamma\left(\frac{1}{4} + i\frac{t}{2}\right), \end{aligned}$$

and  $\{t_\nu\}$  be the sequence defined by the equation (see [4], p. 221)

$$(1.2) \quad \vartheta(t_\nu) = \pi\nu, \quad \nu = 1, 2, \dots$$

In the paper [2] we have obtained with the help of the Riemann-Siegel formula (see [4], p. 220)

$$(1.3) \quad Z(t) = 2 \sum_{n \leq \sqrt{\bar{t}}} \frac{1}{\sqrt{n}} \cos(\vartheta - t \ln n) + \mathcal{O}(t^{-1/4}), \quad \bar{t} = \frac{T}{2\pi},$$

the following formulae

$$(1.4) \quad \begin{aligned} \sum_{T \leq t_\nu \leq T+H} Z(t_\nu) Z(t_\nu + \bar{\tau}_k) &\sim -\frac{1}{(4k+3)\pi^2} H \ln^2 \frac{T}{2\pi}, \\ \sum_{T \leq t_\nu \leq T+H} Z(t_\nu) Z(t_\nu + \bar{\bar{\tau}}_k) &\sim \frac{1}{(4k+1)\pi^2} H \ln^2 \frac{T}{2\pi}, \quad T \rightarrow \infty, \end{aligned}$$

where

$$(1.5) \quad \begin{aligned} \bar{\tau}_k &= \frac{(4k+3)\pi}{\ln \frac{T}{2\pi}}, \quad \bar{\bar{\tau}}_k = \frac{(4k+1)\pi}{\ln \frac{T}{2\pi}}, \\ k &= 0, 1, \dots, K_0(T) = \mathcal{O}(1), \quad H = \sqrt{T} \ln T. \end{aligned}$$

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*Key words and phrases.* Riemann zeta-function.

*Remark 1.* The autocorrelative sum

$$(1.6) \quad \sum_{T \leq t\nu \leq T+H} Z(t\nu)Z(t\nu + \tau_k)$$

is oscillatory on the segment of the arithmetic sequence

$$\bar{\tau}_0, \bar{\tau}_0, \bar{\tau}_1, \bar{\tau}_1, \dots, \bar{\tau}_{K_0}, \bar{\tau}_{K_0}.$$

as it follows by (1.4).

In this paper:

- (a) we use the above mentioned oscillatory behavior of the sum (1.6) in order to construe new kind (in the theory of the Riemann  $Z(t)$ -function) of quasi-orthonormal system of vectors;
- (b) we obtain, on basis of (a), a *microscopic* interpolation formula for the function  $Z(t)$  on a set of segments each of them has the length

$$(1.7) \quad < A \frac{\{\psi(T)\}^\epsilon}{\ln T} \rightarrow 0 \quad \text{as } T \rightarrow \infty,$$

where  $\psi(T)$  is an arbitrary fixed function of the type

$$\ln_3 T, \ln_4 T, \dots; \quad \ln_3 T = \ln \ln \ln T, \dots$$

## 2. RESULT

2.1. The following Theorem holds true.

**Theorem.** The formula

$$(2.1) \quad \begin{aligned} {}^*Z(h_\nu) &= \frac{4}{\pi} \sum_{p=1}^L \frac{(-1)^{p+1}}{2p-1} \frac{{}^*Z(h_\nu + \bar{\tau}_p) + {}^*Z(h_\nu - \bar{\tau}_p)}{2} + \\ &+ \mathcal{O} \left\{ [\psi(T)]^{-\epsilon/4} \right\}, \quad L = [(\psi(T))^\epsilon], \end{aligned}$$

where  $[\dots]$  stands for the integer part, and

$$(2.2) \quad h_\nu = t_\nu + \frac{\pi}{\ln \frac{T}{2\pi}}, \quad \bar{\tau}_p = \frac{(2p-1)\pi}{\ln \frac{T}{2\pi}}, \quad {}^*Z(t) = \frac{Z(t)}{\sqrt{\ln \frac{T}{2\pi}}}$$

holds true for the values

$$h_\nu \in [T, T + \bar{H}]$$

of the order

$$(2.3) \quad \sim \frac{1}{2\pi} \bar{H} \ln \frac{T}{2\pi}, \quad T \rightarrow \infty; \quad \bar{H} = \sqrt{T} \psi(T),$$

i. e. for *the almost all*  $h_\nu$ .

*Remark 2.* The result of our Theorem may be expressed, from the viewpoint of the theory of interpolation, as follows. If we assume the values

$$\{{}^*Z(h_\nu \pm \bar{\tau}_p)\}_{p=1}^L$$

are given, then the formula (2.1) expresses an approximation of the unknown value

$${}^*Z(h_\nu).$$

*Remark 3.* For the discrete set of arguments in (2.1)

$$\{h_\nu - \bar{\tau}_L, h_\nu - \bar{\tau}_{L-1}, \dots, h_\nu, h_\nu + \bar{\tau}_1, \dots, h_\nu + \bar{\tau}_L\}$$

we have (see (2.2))

$$(2.4) \quad h_\nu + \bar{\tau}_L - (h_\nu - \bar{\tau}_L) = 2\bar{\tau}_L < A \frac{(\ln_3 T)^\epsilon}{\ln T} \rightarrow 0, \quad T \rightarrow \infty$$

(comp. (1.7)). Consequently, from (2.4) the title *microscopic* interpolation for (2.1) follows.

2.2. The following Lemma is a basis for the proof of our Theorem.

**Lemma.** If

$$(2.5) \quad \tau', \tau'' = \mathcal{O}\left(\frac{\psi^\epsilon}{\ln T}\right),$$

then we have the following formulae

$$(2.6) \quad \begin{aligned} & \sum_{T \leq t_\nu \leq T + \bar{H}} Z(t_\nu + \tau') Z(t_\nu + \tau'') = \\ & = \frac{1}{2\pi} \frac{\sin\{(\tau'' - \tau') \ln P_0\}}{(\tau'' - \tau') \ln P_0} \bar{H} \ln^2 \frac{T}{2\pi} + \mathcal{O}(\sqrt{T} \ln^2 T), \quad \tau' \neq \tau'', \end{aligned}$$

and

$$(2.7) \quad \sum_{T \leq t_\nu \leq T + \bar{H}} Z^2(t_\nu + \tau') = \frac{1}{2\pi} \bar{H} \ln^2 \frac{T}{2\pi} + \mathcal{O}(\sqrt{T} \ln^2 T),$$

where

$$P_0 = \sqrt{\frac{T}{2\pi}},$$

and these formulae are uniform in  $\tau', \tau''$  (see the condition (2.5)).

This Lemma follows from [2], (10), (11) in the case

$$\begin{aligned} \tau & \rightarrow \tau'' - \tau', \quad \tau = \mathcal{O}\left(\frac{1}{\ln T}\right) \rightarrow \tau', \quad \tau'' = \mathcal{O}\left(\frac{\psi^\epsilon}{\ln T}\right), \\ H & \rightarrow \bar{H} = \sqrt{T} \psi(T). \end{aligned}$$

Next parts of this paper are ordered as follows. We define:

- (a) certain quasi-orthonormal system of vectors,
- (b) an analogue of the Fourier coefficients for this case and, consequently, the asymptotic Fourier coefficients,
- (c) an analogue of the trigonometric polynomial related with our vectors,
- (d) corresponding mean square deviation.

After completion this program we prove the Theorem.

### 3. QUASI-ORTHONORMAL SYSTEM OF VECTORS

Since (see [1], (23))

$$Q = Q(T, \bar{H}) = \sum_{T \leq t_\nu \leq T + \bar{H}} 1 = \frac{1}{2\pi} \bar{H} \ln \frac{T}{2\pi} + \mathcal{O}\left(\frac{\bar{H}^2}{T}\right),$$

then (see (2.3))

$$(3.1) \quad Q \sim \frac{1}{2\pi} \bar{H} \ln \frac{T}{2\pi}; \quad \bar{H} = \sqrt{T} \psi(T).$$

Next, we have in the case

$$(3.2) \quad \tau_p = \frac{2\pi}{\ln \frac{T}{2\pi}} p, \quad p = -L+1, \dots, -1, 0, 1, \dots, L$$

that (see (2.5), (2.6), (3.2))

$$(3.3) \quad \sum_{T \leq t_\nu \leq T+\bar{H}} Z^2(t_\nu + \tau_p) = \frac{1}{2\pi} \bar{H} \ln^2 \frac{T}{2\pi} + \mathcal{O}(\sqrt{T} \ln^2 T),$$

and

$$(3.4) \quad \sum_{T \leq t_\nu \leq T+\bar{H}} Z(t_\nu + \tau_p) Z(t_\nu + \tau_{p'}) = \mathcal{O}(\sqrt{T} \ln^2 T), \quad p \neq p'.$$

Consequently, we obtain (see (2.2), (3.1) – (3.4)) the following formula

$$(3.5) \quad \begin{aligned} & \frac{1}{Q} \sum_{T \leq t_\nu \leq T+\bar{H}} Z^*(t_\nu + \tau_p) Z^*(t_\nu + \tau_{p'}) = \\ & = \begin{cases} \mathcal{O}(\frac{1}{\psi}) & , \quad p \neq p', \\ 1 + \mathcal{O}(\frac{1}{\psi}) & , \quad p = p'. \end{cases} \end{aligned}$$

*Remark 4.* We will call the property (3.5) of the system of vectors

$$(3.6) \quad \{Z^*(t_\nu + \tau_p)\}, \quad t_\nu \in [T, T + \bar{H}]; \quad p = -L+1, \dots, -1, 0, 1, \dots, L$$

as *quasi-orthonormality*.

#### 4. ANALOGUE OF FOURIER COEFFICIENTS AND THE CLASSICAL LEIBNITZ SERIES

We define following numbers

$$(4.1) \quad \begin{aligned} A_p &= \frac{1}{Q} \sum_{T \leq t_\nu \leq T+\bar{H}} Z^*(t_\nu + \tau^0) Z^*(t_\nu + \tau_p), \\ p &= -L+1, \dots, -1, 0, 1, \dots, L \end{aligned}$$

as an analogue of the Fourier coefficients of the vector

$$(4.2) \quad Z^*(t_\nu + \tau^0), \quad T \leq t_\nu \leq T + \bar{H}, \quad \tau^0 = \frac{\pi}{\ln \frac{T}{2\pi}}.$$

Since (see (3.2), (4.2))

$$(4.3) \quad \tau_p - \tau^0 = \frac{(2p-1)\pi}{\ln \frac{T}{2\pi}} = \left(\pi p - \frac{\pi}{2}\right) \frac{1}{\ln P_0}; \quad P_0 = \sqrt{\frac{T}{2\pi}},$$

then we have (comp. (2.6))

$$(4.4) \quad \frac{\sin\{(\tau_p - \tau^0) \ln P_0\}}{(\tau_p - \tau^0) \ln P_0} = \frac{\pi}{2} \frac{(-1)^{p+1}}{2p-1}.$$

Consequently, we have from (2.6) by (3.1), (4.1), (4.4) that

$$(4.5) \quad A_p = \frac{2}{\pi} \frac{(-1)^{p+1}}{2p-1} + \mathcal{O}\left(\frac{1}{\psi}\right).$$

*Remark 5.* It is natural to call the numbers

$$(4.6) \quad \bar{A}_p = \frac{2}{\pi} \frac{(-1)^{p+1}}{2p-1}, \quad p = -L+1, \dots, -1, 0, 1, \dots, L$$

as the asymptotic Fourier coefficients of the vector (2.4).

*Remark 6.* Let us point out the presence of the members of the classical Leibnitz series

$$\sum_{p=1}^{\infty} \frac{2}{\pi} \frac{(-1)^{p+1}}{2p-1} = \frac{1}{2}$$

in the notion of the asymptotic Fourier coefficients.

## 5. ANALOGUE OF THE TRIGONOMETRIC POLYNOMIAL

Finally, we will define an analogue of the trigonometric polynomial  $P_{2L}$ , that corresponds to our quasi-orthonormal system (3.6) by the following way

$$(5.1) \quad P_{2L}[Z^*(t_\nu + \tau^0)] = \sum_{p=-L+1}^L \bar{A}_p Z^*(t_\nu + \tau_p).$$

Since from (4.6) follows that

$$(5.2) \quad \bar{A}_p = \bar{A}_{-p+1},$$

then we have from (5.1) by (4.6), (5.2)

$$\begin{aligned} P_{2L} &= \sum_{p=1}^L \bar{A}_p Z^*(t_\nu + \tau_p) + \sum_{p=1}^L \bar{A}_{1-p} Z^*(t_\nu + \tau_{1-p}) = \\ (5.3) \quad &= \sum_{p=1}^L \bar{A}_p \{Z^*(t_\nu + \tau_p) + Z^*(t_\nu + \tau_{1-p})\} = \\ &= \frac{2}{\pi} \sum_{p=1}^L \frac{(-1)^{p+1}}{2p-1} \{Z^*(t_\nu + \tau_p) + Z^*(t_\nu + \tau_{1-p})\}. \end{aligned}$$

## 6. MEAN SQUARE DEVIATION AND THE CLASSICAL EULER SERIES

6.1. First of all, from the Euler series by (2.1) we obtain

$$\begin{aligned} (6.1) \quad \frac{\pi^2}{8} &= \sum_{n=1}^{\infty} \frac{1}{(2p-1)^2} = \sum_{p=1}^L \frac{1}{(2p-1)^2} + \sum_{p=L+1}^{\infty} \frac{1}{(2p-1)^2} = \\ &= \sum_{p=1}^L \frac{1}{(2p-1)^2} + \mathcal{O}\left(\frac{1}{L}\right) = \sum_{p=1}^L \frac{1}{(2p-1)^2} + \mathcal{O}\left(\frac{1}{\psi^\epsilon}\right). \end{aligned}$$

Since (see (4.6))

$$(6.2) \quad \bar{A}_p^2 = \frac{4}{\pi^2} \frac{1}{(2p-1)^2},$$

then we have (see (5.2), (6.1), (6.2))

$$\begin{aligned}
 (6.3) \quad \sum_{p=-L+1}^L \bar{A}_p^2 &= \frac{4}{\pi^2} \sum_{p=-L+1}^L \frac{1}{(2p-1)^2} = \frac{8}{\pi^2} \sum_{p=1}^L \frac{1}{(2p-1)^2} = \\
 &= \frac{8}{\pi^2} \left\{ \frac{\pi^2}{8} + \mathcal{O}\left(\frac{1}{\psi^\epsilon}\right) \right\} = 1 + \mathcal{O}\left(\frac{1}{\psi^\epsilon}\right),
 \end{aligned}$$

and, of course,

$$(6.4) \quad \sum_{p,q=-L+1}^L |\bar{A}_p \bar{A}_q| = \mathcal{O}(L^2) = \mathcal{O}(\psi^{2\epsilon}).$$

6.2. We define discrete mean square deviation  $\Delta$  as follows

$$(6.5) \quad \Delta^2 = \frac{1}{Q} \sum_{T \leq t_\nu \leq T+\bar{H}} \{Z^*(t_\nu u + \tau^0) - P_{2L}\}^2.$$

Consequently, we have

$$\begin{aligned}
 (6.6) \quad \Delta^2 &= \frac{1}{Q} \sum_{(t_\nu)} Z^*(t_\nu + \tau^0)^2 - \frac{2}{Q} \sum_{(t_\nu)} Z^*(t_\nu + \tau^0) P_{2L} + \\
 &+ \frac{1}{Q} \sum_{(t_\nu)} P_{2L}^2 = w_1 + w_2 + w_3.
 \end{aligned}$$

From (2.7),  $\tau' = \tau''$ , we obtain immediately (see (2.2), (2.5), (3.1), (4.2)) that

$$(6.7) \quad w_1 = 1 + \mathcal{O}\left(\frac{1}{\psi}\right).$$

6.3. Next, we obtain from (5.1), (6.6) by (4.1), (4.5), (4.6), (6.3)

$$\begin{aligned}
 (6.8) \quad w_2 &= -\frac{2}{Q} \sum_{(t_\nu)} Z^*(t_\nu + \tau^0) \sum_{p=-L+1}^L \bar{A}_p Z^*(t_p + \tau_p) = \\
 &= -2 \sum_{p=-L+1}^L \bar{A}_p \frac{1}{Q} \sum_{(t_\nu)} Z^*(t_\nu + \tau^0) Z^*(t_\nu + \tau_p) = \\
 &= -2 \sum_{p=-L+1}^L \bar{A}_p A_p = -2 \sum_{p=-L+1}^L \bar{A}_p^2 + \mathcal{O}\left(\frac{1}{\psi} \sum_{p=-L+1}^L 1\right) = \\
 &= -2 + \mathcal{O}\left(\frac{1}{\psi^\epsilon}\right) + \mathcal{O}\left(\frac{1}{\psi^{1-\epsilon}}\right) = \\
 &= -2 + \mathcal{O}\left(\frac{1}{\psi^\epsilon}\right).
 \end{aligned}$$

6.4. Once again we have (see (3.5), (5.1), (6.3), (6.4))

$$\begin{aligned}
 w_3 &= \sum_{p,q=-L+1}^L \bar{A}_p \bar{A}_q \frac{1}{Q} \sum_{(t_\nu)}^* Z(t_\nu + \tau_p) Z(t_\nu + \tau_q) = \\
 (6.9) \quad &= \sum_{p=-L+1}^L \bar{A}_p^2 \left\{ 1 + \mathcal{O}\left(\frac{1}{\psi}\right) \right\} + \mathcal{O}\left\{ \frac{1}{\psi} \sum_{p,q=-L+1}^L |\bar{A}_p \bar{A}_q| \right\} = \\
 &= 1 + \mathcal{O}\left(\frac{1}{\psi^\epsilon}\right) + \mathcal{O}\left(\frac{L^2}{\psi}\right) = \\
 &= 1 + \mathcal{O}\left(\frac{1}{\psi^\epsilon}\right).
 \end{aligned}$$

Consequently, we obtain from (6.6) by (6.7) – (6.9) the following estimate

$$(6.10) \quad \Delta^2 < \frac{A}{\{\psi(T)\}^\epsilon}.$$

## 7. FINALISATION OF THE PROOF OF THE THEOREM

Let  $R(T)$  denote the number of points

$$\bar{t}_\nu \in [T, T + \bar{H}]$$

that fulfill the inequality

$$(7.1) \quad |\bar{Z}(\bar{t}_\nu + \tau^0) - P_{2L}[\bar{Z}(\bar{t}_\nu + \tau^0)]| \geq \frac{1}{\psi^{\epsilon/4}}.$$

Our assertion is that

$$(7.2) \quad R(T) = o(\bar{H} \ln T), \quad T \rightarrow \infty.$$

Namely, if

$$(7.3) \quad R(T) > B \bar{H} \ln T$$

for every fixed  $B > 0$ , then we have from (6.10) by (6.5), (7.1), (7.3) that

$$\begin{aligned}
 \frac{A}{\psi^\epsilon} &\geq \frac{1}{Q} \sum_{(t_\nu)}^* \{ \bar{Z}(\bar{t}_\nu + \tau^0) - P_{2L}[\bar{Z}(\bar{t}_\nu + \tau^0)] \}^2 > \\
 &> \frac{B \bar{H} \ln T}{Q} \frac{1}{\psi^{\epsilon/2}} > \frac{C}{\psi^{\epsilon/2}}, \quad T \rightarrow \infty,
 \end{aligned}$$

i. e. we have contradiction. Hence, our assertion (7.2) is true. Consequently, if  $Q_1(T)$  denotes the number of the points

$$t_\nu \in [T, T + \bar{H}]$$

that fulfill the inequality

$$(7.4) \quad |\bar{Z}(t_\nu + \tau^0) - P_{2L}[\bar{Z}(t_\nu + \tau^0)]| < \frac{1}{\psi^{\epsilon/4}},$$

then (see (3.1), (7.2))

$$(7.5) \quad Q_1(T, \bar{H}) \sim \frac{1}{2\pi} \bar{H} \ln \frac{T}{2\pi}, \quad T \rightarrow \infty,$$

i. e. the assertion of Theorem is true if we use the following notation

$$t_\nu + \tau^0 = h_\nu, \quad \tau_p - \tau^0 = \bar{\tau}_p, \quad \tau_{1-p} - \tau^0 = -\bar{\tau}_p.$$

APPENDIX A. NEW ALGORITHM FOR CONSTRUCTION OF AN ASYMPTOTIC  
SOLUTION TO A DIOPHANTINE EQUATION WITH LEIBNITZ  
COEFFICIENTS

First of all we express our formula (2.1) in the following form

$$(A.1) \quad -\frac{\pi}{4} \bar{Z}(h_\nu) + \sum_{p=1}^L \frac{(-1)^{p+1}}{2p-1} \frac{\bar{Z}(h_\nu + \bar{\tau}_p) + \bar{Z}(h_\nu - \bar{\tau}_p)}{2} = \\ = \mathcal{O}(\psi^{-\epsilon/4}) \sim 0, \quad T \rightarrow \infty.$$

Next, we introduce (on this basis) the following Diophantine equation

$$(A.2) \quad \sum_{p=0}^L a_p X_p = 0,$$

where the numbers

$$(A.3) \quad a_0 = -\frac{\pi}{4}, \quad a_p = \frac{(-1)^{p+1}}{2p-1}, \quad p = 1, 2, \dots, L; \\ a_0 \in \mathbb{R} \setminus \mathbb{Q}, \quad a_p \in \mathbb{Q}$$

are Leibnitz coefficients, i. e. the members of the classical Leibnitz series

$$-\frac{\pi}{4} + \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{2p-1} = 0.$$

Hence, from Theorem we have that for all sufficiently big

$$T > 0$$

there is unbounded set of asymptotic solutions

$$(A.4) \quad X_0 = \bar{Z}(h_\nu), \quad X_p = \frac{\bar{Z}(h_\nu + \bar{\tau}_p) + \bar{Z}(h_\nu - \bar{\tau}_p)}{2}, \\ p = 1, \dots, L, \quad (L = [\{\psi\}^\epsilon] \rightarrow \infty \quad \text{as} \quad T \rightarrow \infty)$$

of Diophantine equation (A.2). Namely, for the number  $Q_1(T)$  of

$$h_\nu \in [T, T + \bar{H}]$$

that generate asymptotic solutions, we have (see (2.3), (7.5))

$$Q_1(T) \sim \frac{1}{2\pi} \bar{H} \ln \frac{T}{2\pi} = \frac{1}{2\pi} \sqrt{T} \psi(T) \ln \frac{T}{2\pi} \rightarrow \infty \quad \text{as} \quad T \rightarrow \infty.$$

*Remark 7.* It is clear, that from the point of view of this Appendix, the content of this paper may be interpreted as new algorithm how to construct certain set of asymptotic solution (A.4) of the Diophantine equation (A.2) with the Leibnitz coefficients (A.3). In order to attain this result we have used the values of the Riemann  $Z(t)$ -function considering its argument  $t$  belongs to the *microscopic* segment (see (2.4)).

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